The Steady Magnetic Fields

Prepared By

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Agenda

- Intended Learning Outcomes
- Why Study Magnetic Field
- Biot-Savart Law
- Ampere’s Circuital Law
- Magnetic Flux And Magnetic Flux Density
- Maxwell’s Equations for Static Fields
- Gauss’s law for magnetic field
- Magnetic Boundary Conditions
- Magnetic Forces and Inductance
- Magnetic energy density

Intended Learning Outcomes

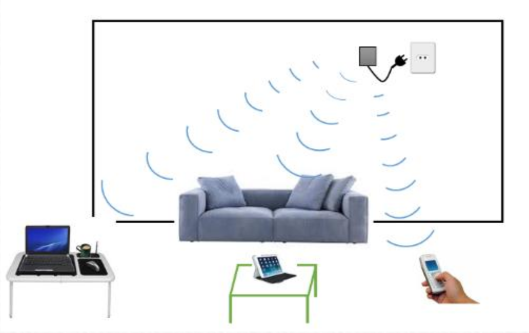
- Define the magnetic field and show how it arises from a current distribution.
- Apply ampere’s circuital law to simplify many problems like determine magnetic field of loop current, solenoid, and toroid.
- Force on a moving point charge and on a filamentary current.
- Force between two filamentary currents.
- Magnetic boundary conditions.
- Self inductance and mutual inductance.
- Magnetic energy density.

Why study magnetic field

Motivations

Complicated powering system

Free and easy WPT



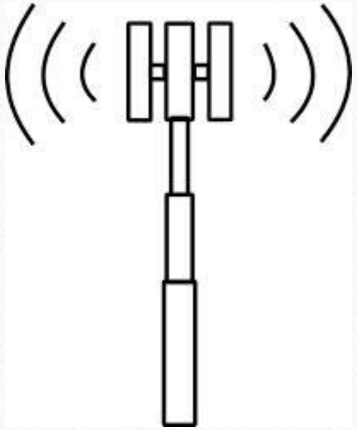
Why study magnetic field

There are two different ways to provide electrical energy wirelessly:

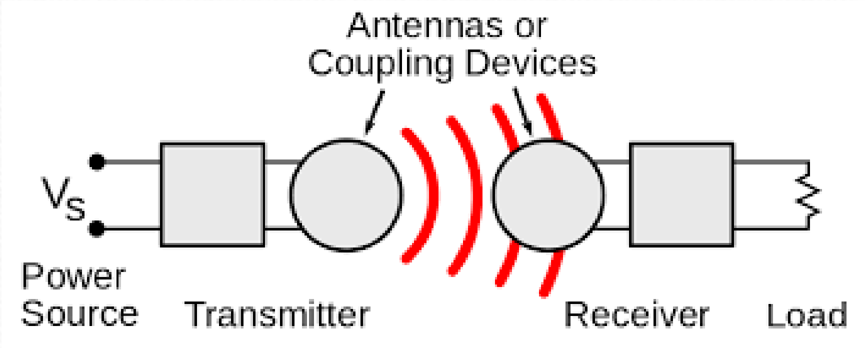
- *Energy harvesting (EH).*
- *Wireless power transmission (WPT).*

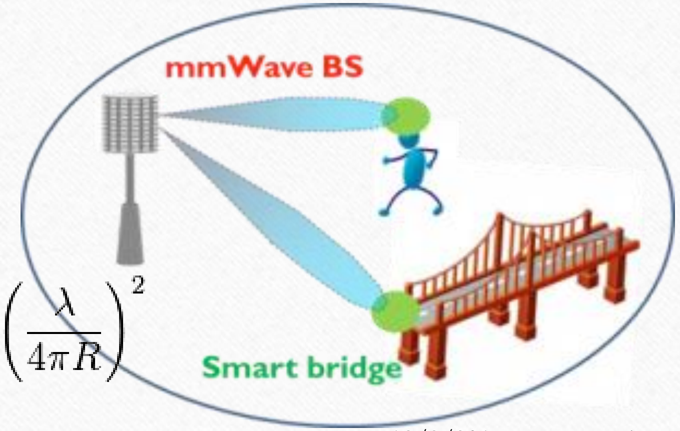
Why study magnetic field

Energy Harvesting



Wireless Power Transfer



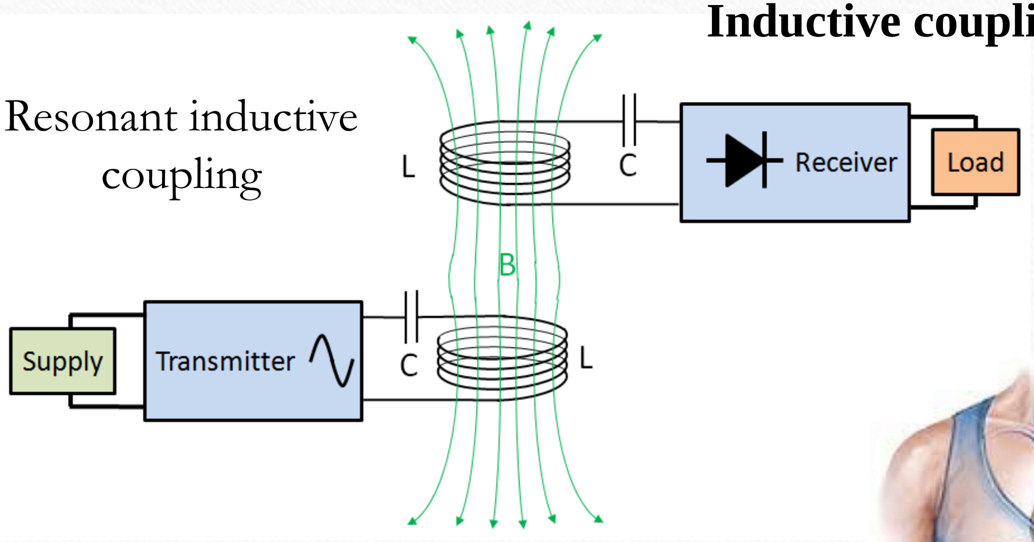


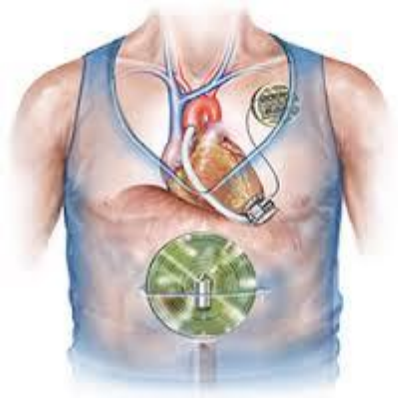
$$\frac{P_r}{P_t} = G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2$$

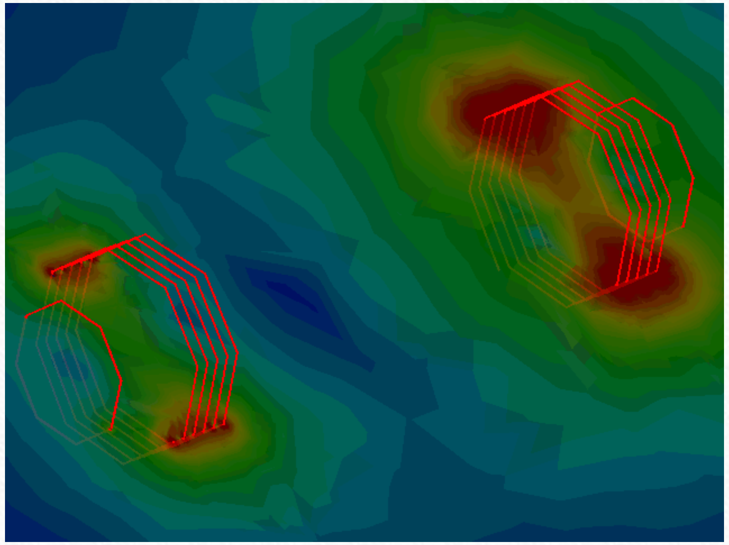
Why study magnetic field

Inductive coupling for WPT

Resonant inductive coupling







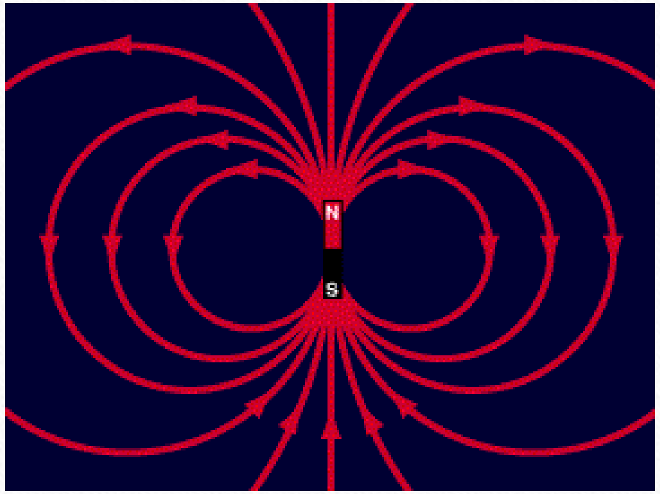
WPT for biomedical applications

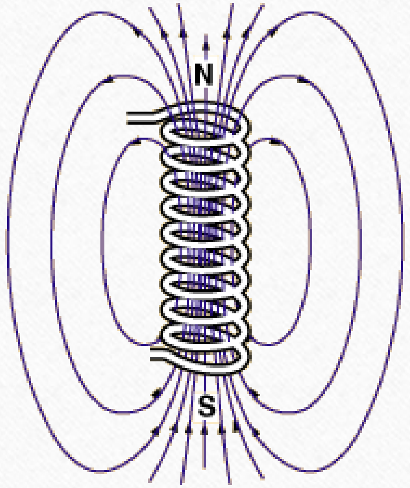
Biot-Savart Law

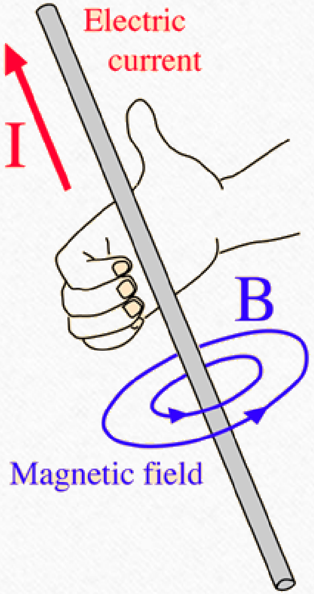
The source of the steady magnetic field may be:

- Permanent magnet, or
- DC current.

In this chapter, we are only concerned the magnetic fields produced by dc currents





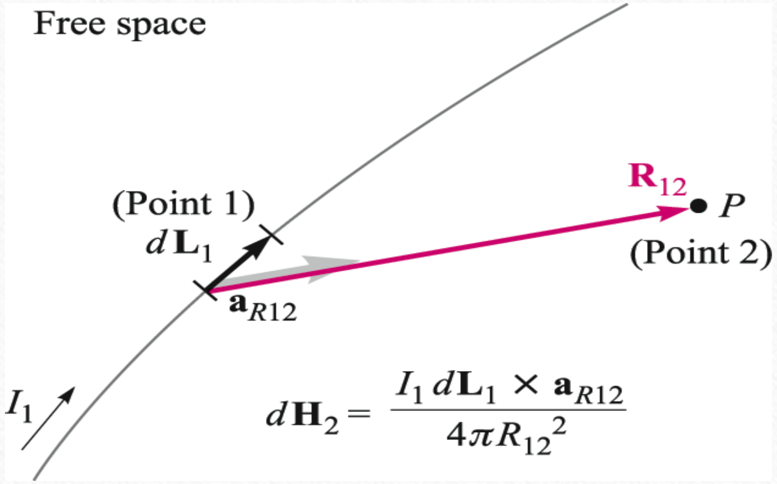


Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, **H**, arising from a "point source" current element of differential length d**L.**

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

The units of **H** are [A/m]

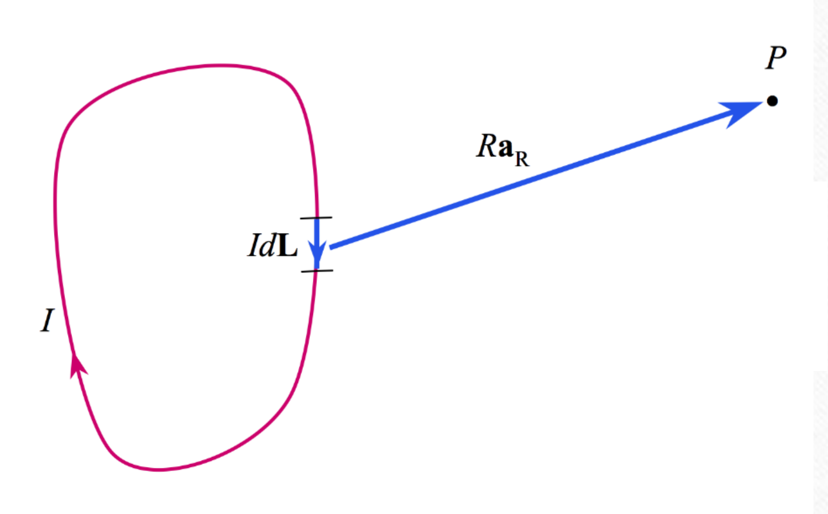


Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page.

Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

Biot-Savart Law



Magnetic Field Arising From a Circulating Current

At point P, the magnetic field associated with the differential current element Id**L** is

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \qquad (1)$$

To determine the total field arising from the closed circuit path, we sum the contributions from the current elements that make up the entire loop, or

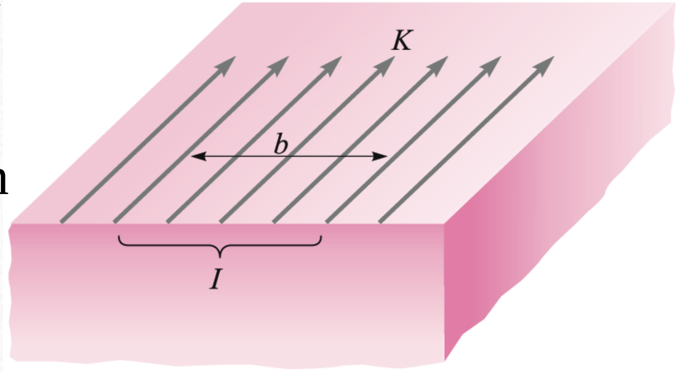
$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}} \qquad (2)$$

Biot-Savart Law

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width b is

$$I = Kb$$



Thus the differential current element I dL, where dL is in the direction of the current, may be expressed in terms of surface current density K or current density J,

$$I\, d\mathbf{L} = \mathbf{K}\, dS = \mathbf{J}\, dv$$

The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2}$$

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

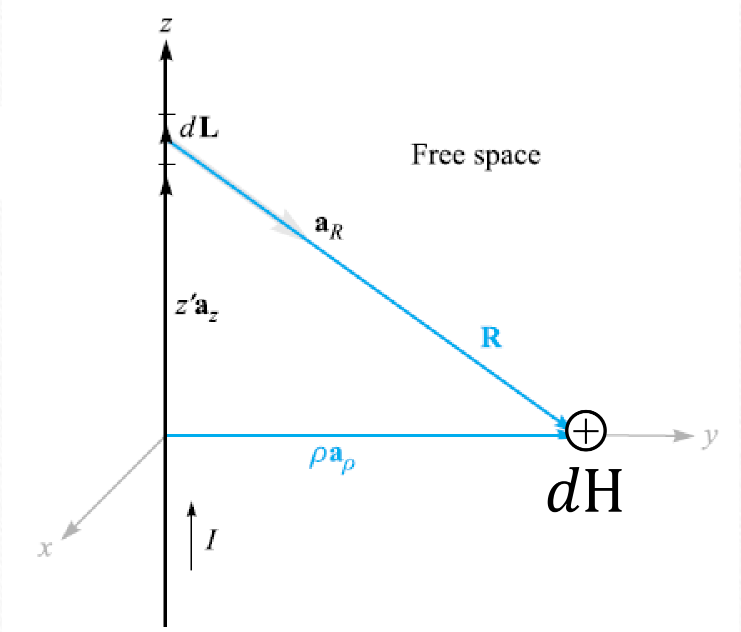
$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2}$$

Biot-Savart Law

Example of the Biot-Savart Law

In this example, we evaluate the magnetic field intensity on the y axis (equivalently in the xy plane) arising from a filament current of infinite length on the z axis.

Using the drawing, we identify:



$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

and so..

$$\mathbf{a}_R = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

so that:

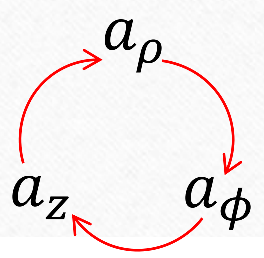
$$d\mathbf{H} = \frac{I\,d\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Biot-Savart Law

Example: continued

We now have:

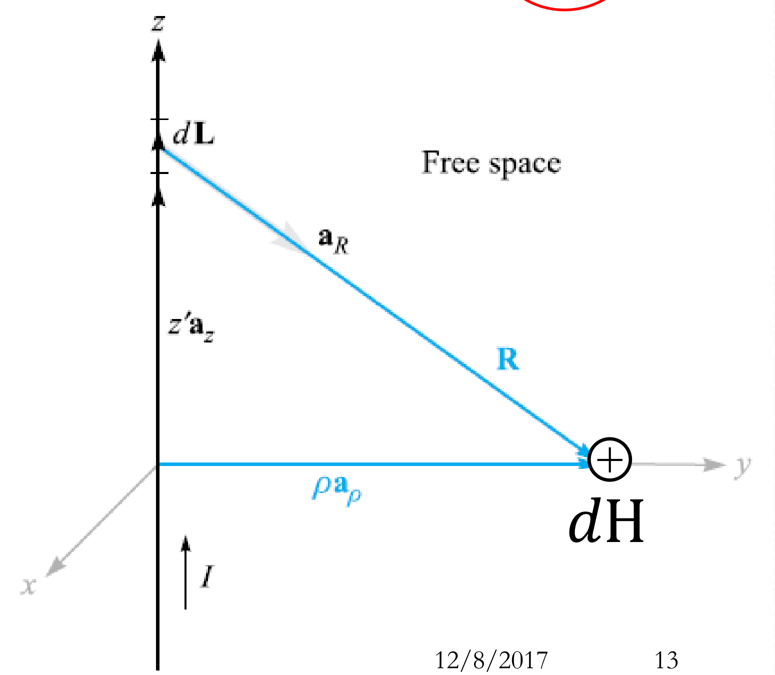
$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$



Integrate this over the entire wire:

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



..after carrying out the cross product

Biot-Savart Law

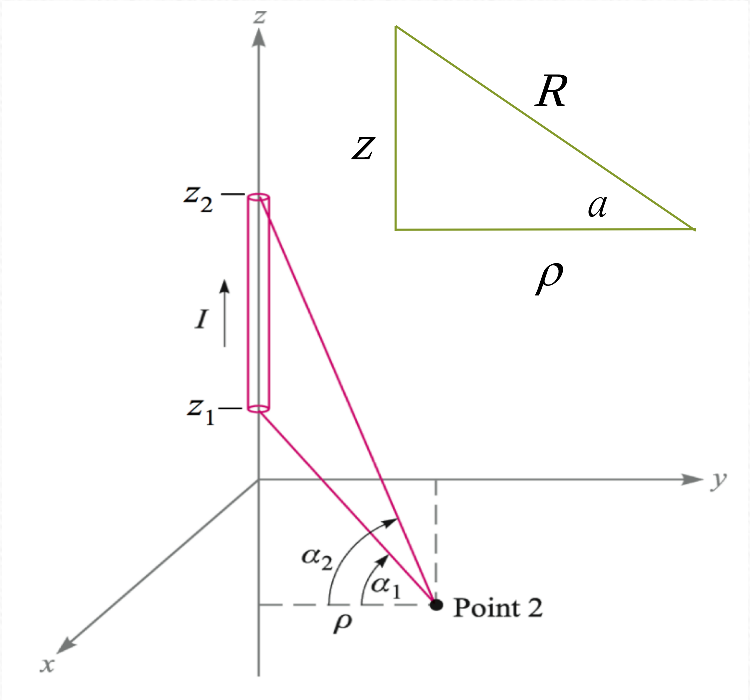
Field arising from a Finite Current Segment

In this case, the field is to be found in the xy plane at Point 2. The Biot-Savart integral is taken over the wire length:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$\overline{\mathrm{H}} = \int_{z_1}^{z_2} \frac{I\rho\, dz}{4\pi\left(\rho^2 + z^2\right)^{3/2}} \overline{a}_\phi$$

$$= \frac{I\rho}{4\pi} \int_{z_1}^{z_2} \frac{dz}{\left(\rho^2 + z^2\right)^{3/2}} \overline{a}_\phi \qquad (3)$$



Using the drawing, $\dfrac{z}{\rho} = \tan\alpha \;\Rightarrow z = \rho\tan\alpha$

$$\therefore dz = \rho \sec^2\alpha \, d\alpha$$

Biot-Savart Law

Example: continued

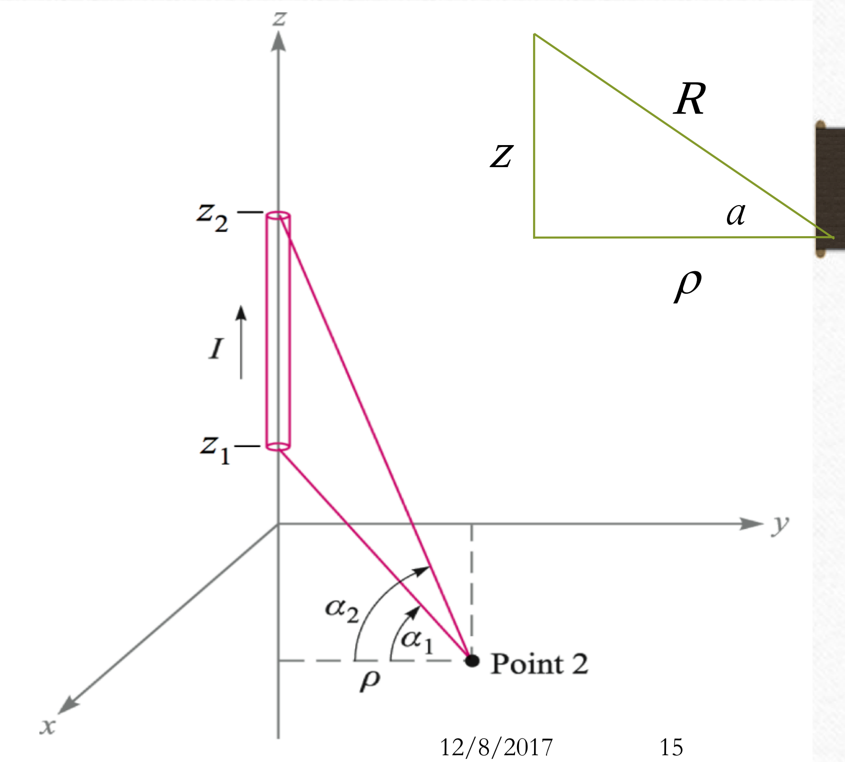
Using the drawing, also $\dfrac{\rho}{R} = \cos\alpha \;\Rightarrow R = \rho\sec\alpha$

$$\because \int \frac{dz}{\left(\rho^2 + z^2\right)^{3/2}} = \int \frac{dz}{R^3} = \int \frac{\rho\sec^2\alpha}{\rho^3\sec^3\alpha} d\alpha$$

$$= \frac{1}{\rho^2}\int_{\alpha_1}^{\alpha_2} \cos\alpha \, d\alpha$$

$$= \frac{1}{\rho^2}\left(\sin\alpha_2 - \sin\alpha_1\right)$$

$$= \frac{1}{\rho^2}\left(\frac{z_2}{\left(\rho^2 + z_2^2\right)^{1/2}} - \frac{z_1}{\left(\rho^2 + z_1^2\right)^{1/2}}\right) \qquad (4)$$



Biot-Savart Law

Example: continued

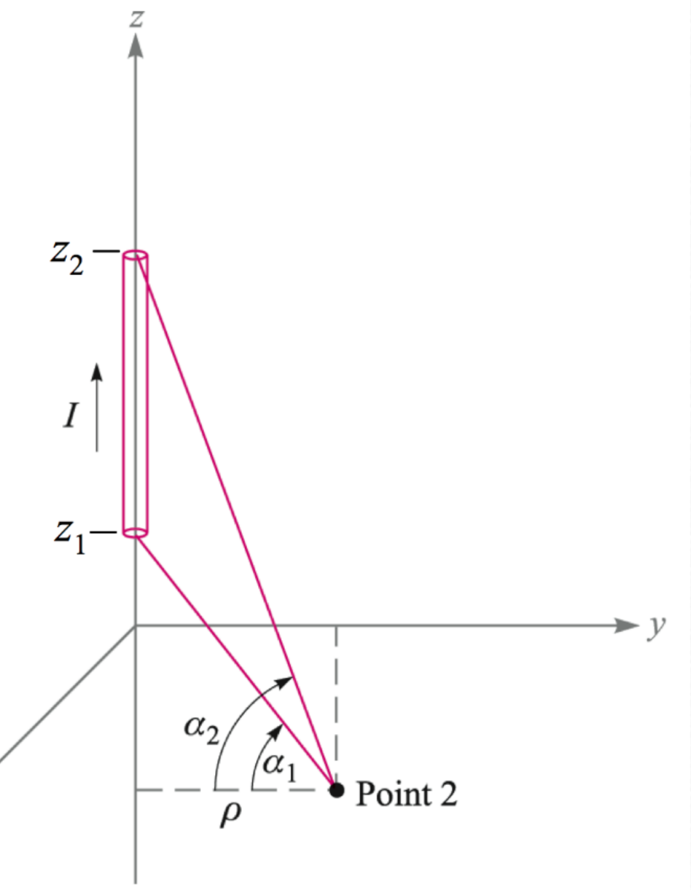
By substitution of (4) in (3)

$$\overline{\mathrm{H}} = \frac{I}{4\pi\rho}\left(\frac{z_2}{\sqrt{\rho^2 + z_2^2}} - \frac{z_1}{\sqrt{\rho^2 + z_1^2}}\right)\overline{\mathrm{a}}_\phi \quad (5)$$

or

$$\overline{\mathrm{H}} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\overline{\mathrm{a}}_\phi \quad (6)$$

If one or both ends are below point 2, then α_1 is or both α_1 and α_2 are negative.



Biot-Savart Law

Example: continued

Special case: for infinite wire → -∞ < Z < ∞

$$\text{because}\quad z = \rho\tan\alpha \qquad \therefore \alpha_1 = -90, \quad \alpha_2 = 90$$

Evaluating the integral in (6):

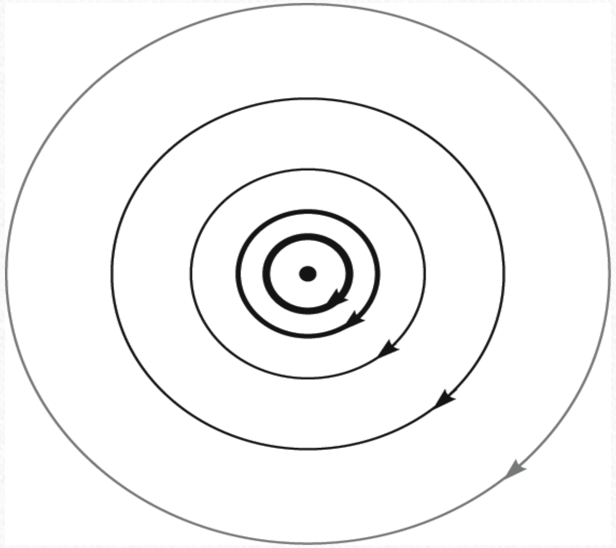
$$\overline{\mathrm{H}} = \frac{I}{4\pi\rho}\left(\sin(90) - \sin(-90)\right)\overline{\mathrm{a}}_\phi$$

we have:

$$= \frac{2I}{4\pi\rho}\overline{\mathrm{a}}_\phi$$

finally:

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi} \qquad (7)$$

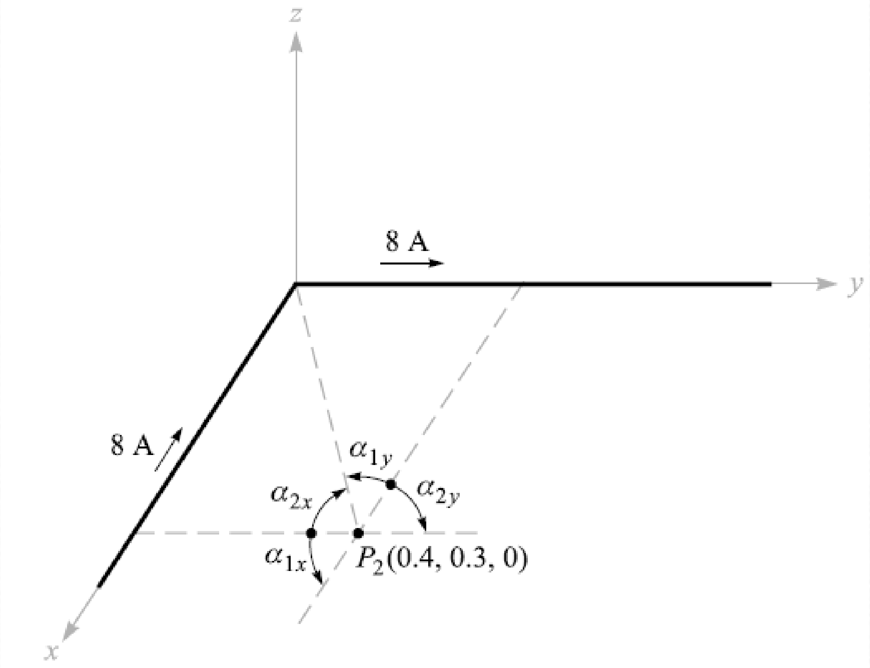


Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the z axis

Biot-Savart Law

Example 8.1:

Determine **H** at $P_2(0.4, 0.3, 0)$ in the field of an **8 A** filamentary current is directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in the figure below.



Solution:

We first consider the semi-infinite current on the x axis → $-\infty < x < 0$

$$\text{because} \;\; x = y \tan \alpha_x$$

$$\therefore \alpha_{1x} = -90, \quad \alpha_{2x} = \tan^{-1}(0.4/0.3)$$

The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$.

Biot-Savart Law

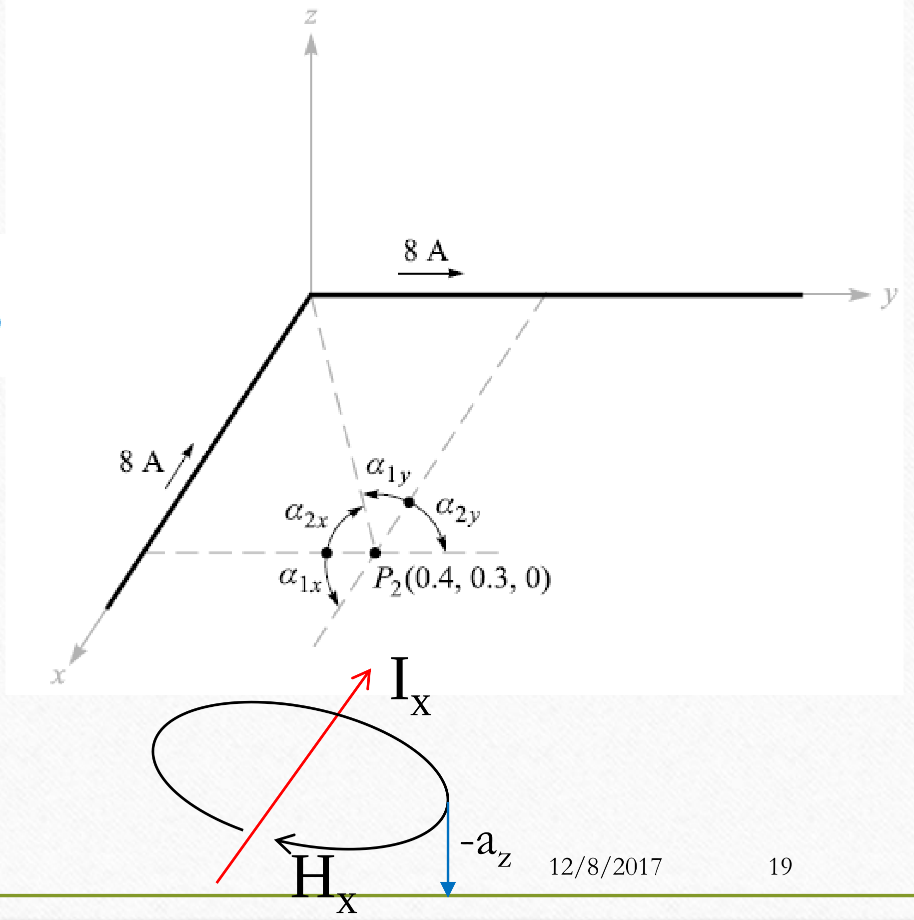
$$\because \overline{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\overline{a}_\phi$$

Thus, this contribution to $\mathbf{H}_2$ is

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector a_ϕ must also be referred to the x axis. We see that it becomes $-a_z$ .Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$



Biot-Savart Law

For the current on the y axis, we have

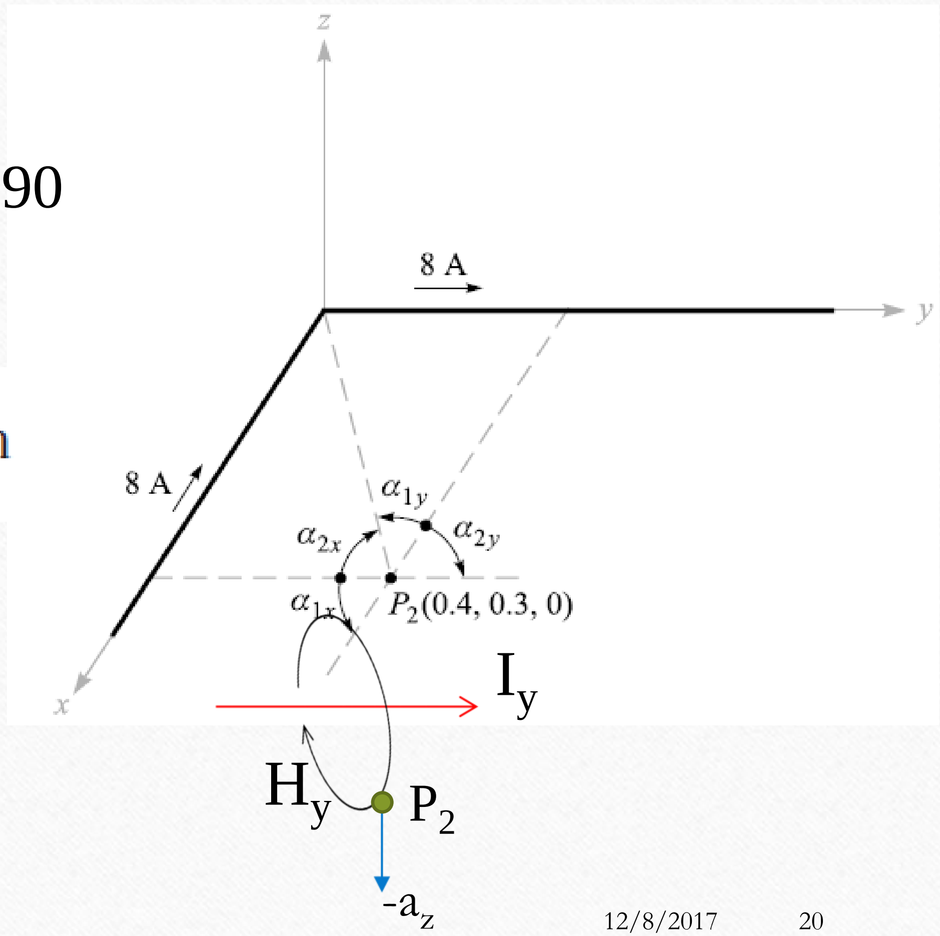
$$\rho_y = 0.4, \alpha_{1y} = -\tan^{-1}(0.3/0.4) = \text{-}36.9^\circ, \text{ and } \alpha_{2y} = 90$$

It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1+\sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \text{ A/m}$$



Home work

Biot-Savart Law

D8.1. Given the following values for P_1, P_2, and $I_1 \Delta L_1$, calculate $\Delta \mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi \mathbf{a}_z \mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi \mathbf{a}_z \mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z$ nA/m

D8.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

Biot-Savart Law

Another Example: Magnetic Field from a Current Loop

Consider a circular current loop of radius ***a*** in the x-y plane, which carries steady current I. We wish to find the magnetic field strength anywhere on the z axis.

We will use the Biot-Savart Law:

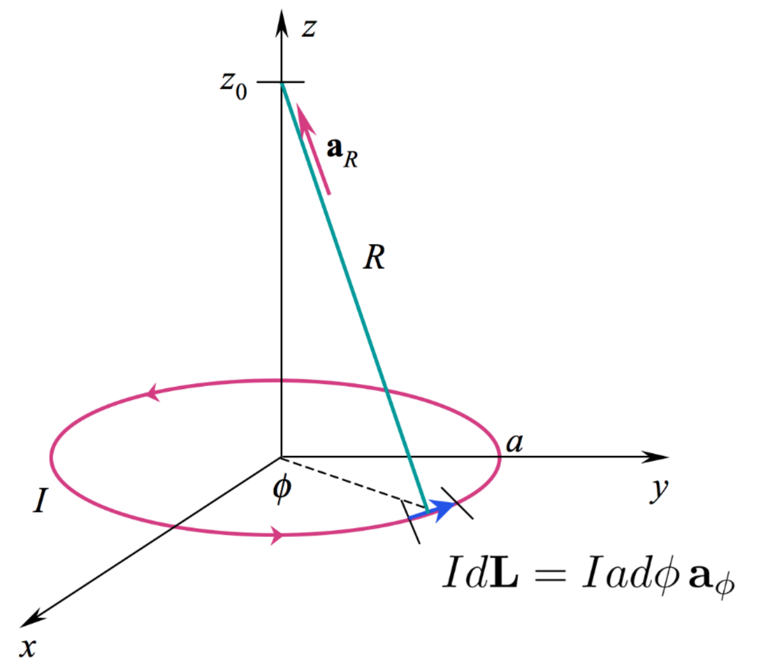
$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where:

$$I d\mathbf{L} = I a d\phi \, \mathbf{a}_\phi$$

$$R = \sqrt{a^2 + z_0^2}$$

$$\mathbf{a}_R = \frac{z_0 \, \mathbf{a}_z - a \, \mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}}$$



Biot-Savart Law

Example continued

Substituting the previous expressions, the Biot-Savart Law becomes:

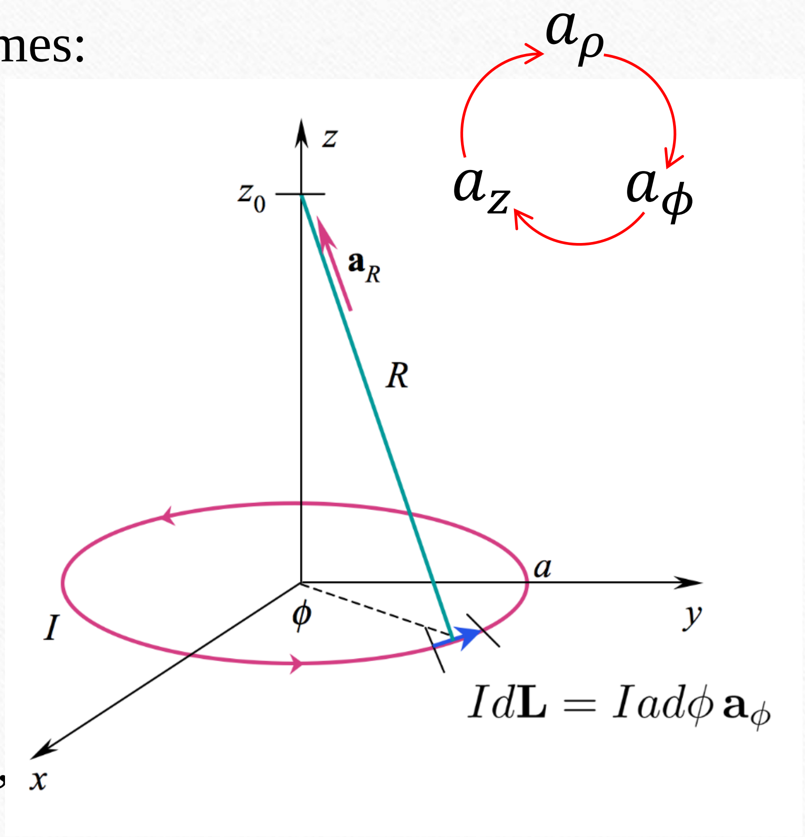
$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z_0\,\mathbf{a}_z - a\,\mathbf{a}_\rho)}{4\pi(a^2 + z_0^2)^{3/2}}$$

carry out the cross products to find:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,(z_0\,\mathbf{a}_\rho + a\,\mathbf{a}_z)}{4\pi(a^2 + z_0^2)^{3/2}}$$

but we must include the angle dependence in the radial unit vector: $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$

with this substitution, the radial component will integrate to zero, meaning that all radial components will cancel on the z axis.

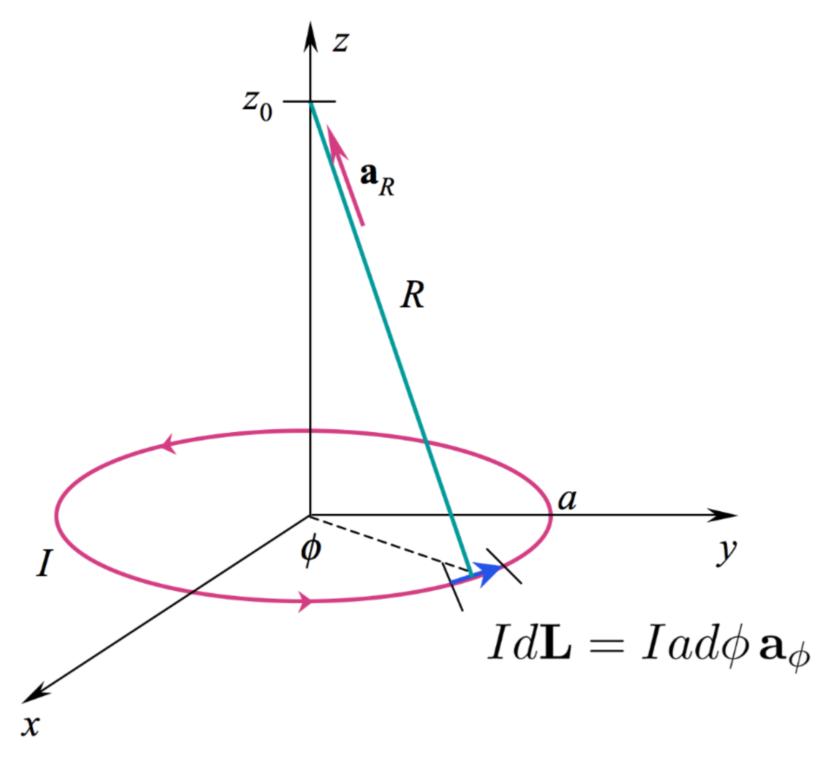


Biot-Savart Law

Example continued

Now, only the z component remains, and the integral evaluates easily:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

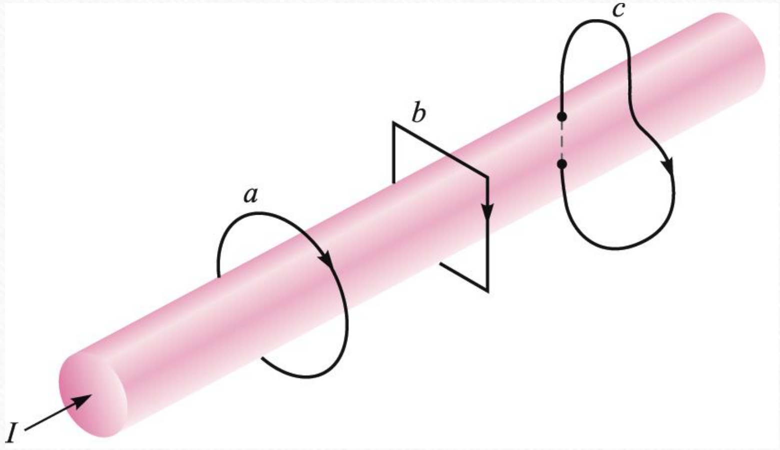


Ampere's Circuital Law

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields.

Ampere's Circuital Law states that the line integral of **H** about any closed path is exactly equal to the direct current enclosed by that path.

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I}$$



In the figure at right, the integral of **H** about closed paths a and b gives the total current I, while the integral over path c gives only that portion of the current that lies within c

Ampere's Circuital Law

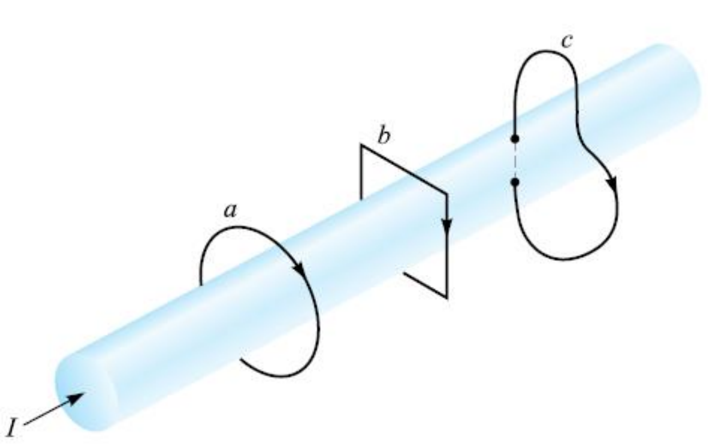
Ampere's Law Applied to a Long Wire

Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (z) axis.

Choosing path a, and integrating **H** around the circle of radius ρ gives the enclosed current, I:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

so that: $\boxed{H_\phi = \frac{I}{2\pi\rho}}$ as before, given in Eq. (7).



Ampere's Circuital Law

Important notes

The Biot-Savart law is resemble the coulomb's law. Both show linear relationship between source and field

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}}$$

Biot-Savart Law

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

Coulomb's Law

Ampere's Circuital law is resemble the gauss's law.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

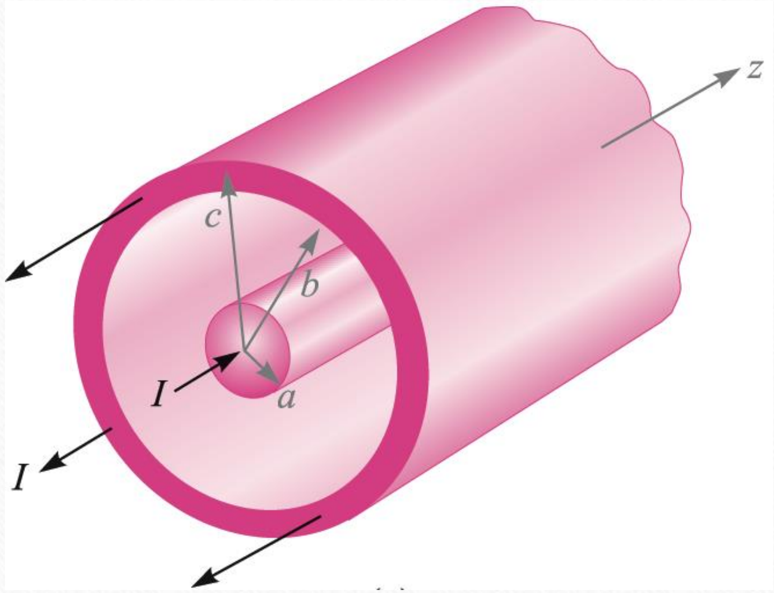
Ampere's Circuital Law

$$\oiint D \cdot ds = Q_{encl}$$

Gauss's Law

Ampere's Circuital Law

Coaxial Transmission Line



In the coax line, we have two concentric solid conductors that carry equal and opposite currents, I.

The line is assumed to be infinitely long, and the circular symmetry suggests that **H** will be entirely ϕ - directed, and will vary only with radius ρ.

Our objective is to find the magnetic field for all values of ρ

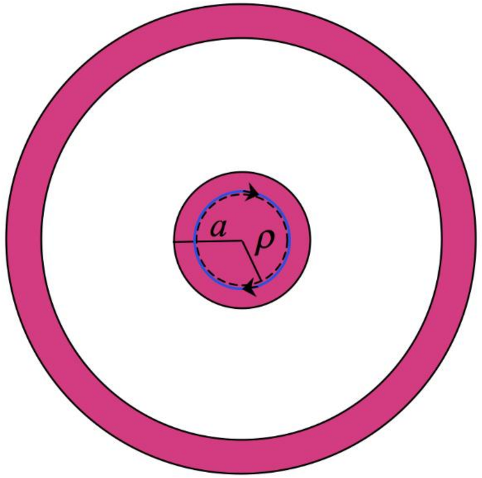
Ampere's Circuital Law

Coaxial Transmission Line

Field Within the Inner Conductor $\boldsymbol{0 < \rho < a}$

With current uniformly distributed inside the conductors, the **H** can be assumed circular everywhere.

Inside the inner conductor, and at radius ρ, we again have:



$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi 2\pi\rho$$

But now, the current enclosed is

$$I_{\text{encl}} = J(\pi\rho^2), \quad \because J = \frac{I}{\pi a^2}$$

$$\therefore I_{\text{encl}} = \frac{I}{\pi a^2}(\pi\rho^2) = \frac{I\rho^2}{a^2}$$

so that $$2\pi\rho H_\phi = I\frac{\rho^2}{a^2}$$

or finally: $$\boxed{H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho < a)} \qquad (8)$$

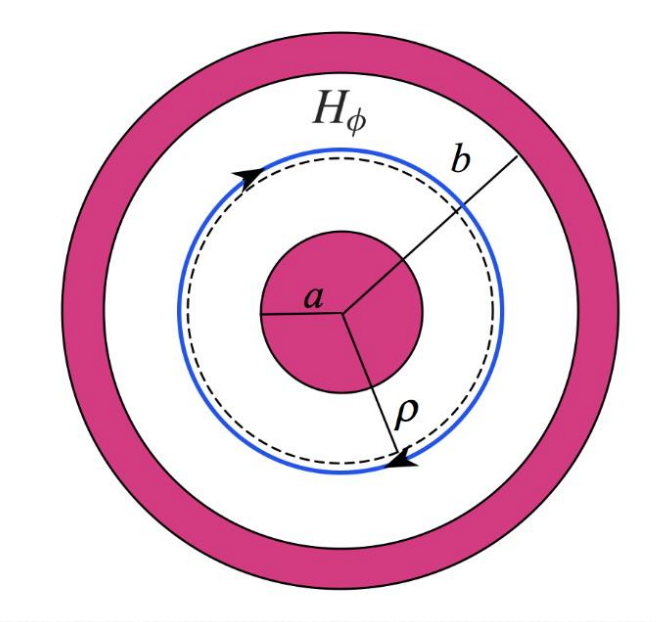
Ampere's Circuital Law

Coaxial Transmission Line

Field Between Conductors

The field between conductors is thus found to be the same as that of filament conductor on the z axis that carries current, I. Specifically:

$$H_\phi = \frac{I}{2\pi\rho} \qquad a < \rho < b \qquad (9)$$

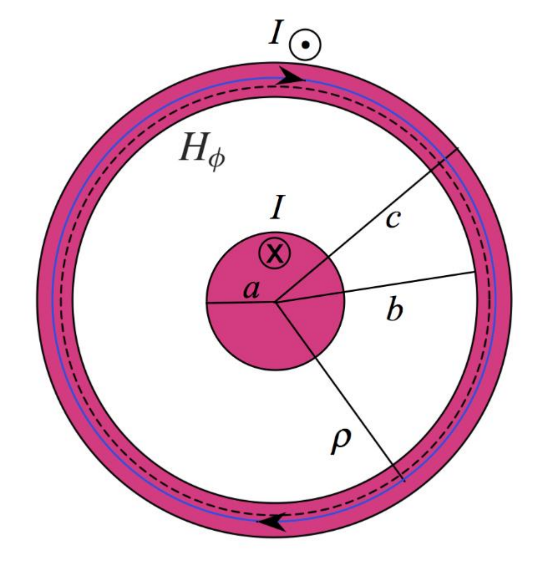


Ampere's Circuital Law

Coaxial Transmission Line

Field Inside the Outer Conductor

Inside the outer conductor, the enclosed current consists of that within the inner conductor plus that portion of the outer conductor current existing at radii less than ρ



$$I_{encl} = I + J_{outer}\left(\pi\rho^2 - \pi b^2\right)$$

$$= I + \left(\frac{-I}{\pi c^2 - \pi b^2}\right)\left(\pi\rho^2 - \pi b^2\right) = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

Ampere's Circuital Law becomes $2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$

..and so finally: $$\boxed{H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)} \qquad (10)$$

Ampere's Circuital Law

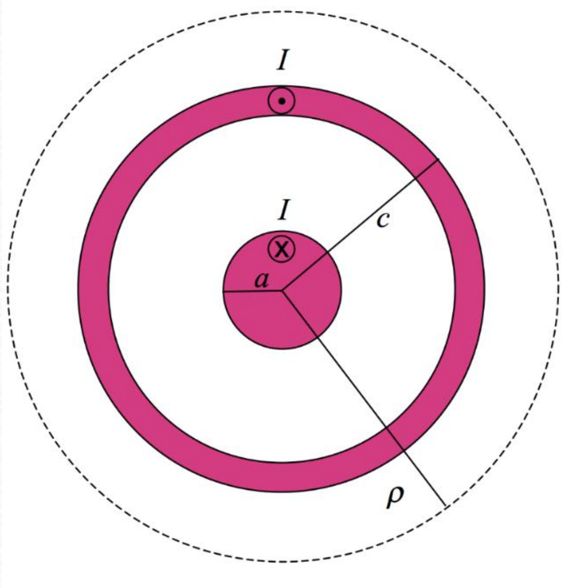
Coaxial Transmission Line

Field Outside Both Conductors

Outside the transmission line, where $\rho > c$, no current is enclosed by the integration path, and so

$$I_{encl} = I + (-I) = 0$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0$$



As the current is uniformly distributed, and since we have circular symmetry, the field would have to be constant over the circular integration path, and so it must be true that:

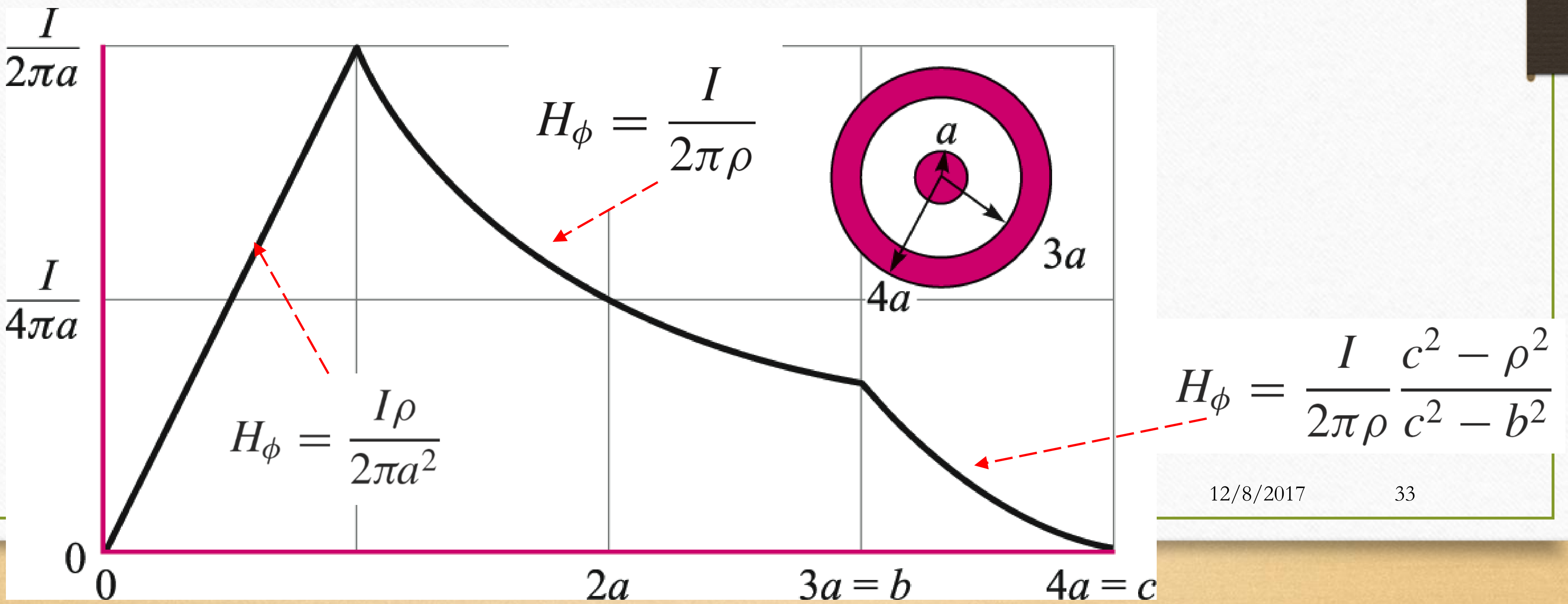
$$\boxed{H_\phi = 0 \quad (\rho > c)} \qquad (11)$$

Ampere's Circuital Law

Coaxial Transmission Line

Magnetic Field Strength as a Function of Radius in the Coax Line

Combining the previous results, and assigning dimensions for a coaxial cable in which $b = 3a$, $c = 4a$. , we find the magnetic-field-strength variation with radius is as shown in the Figure below.



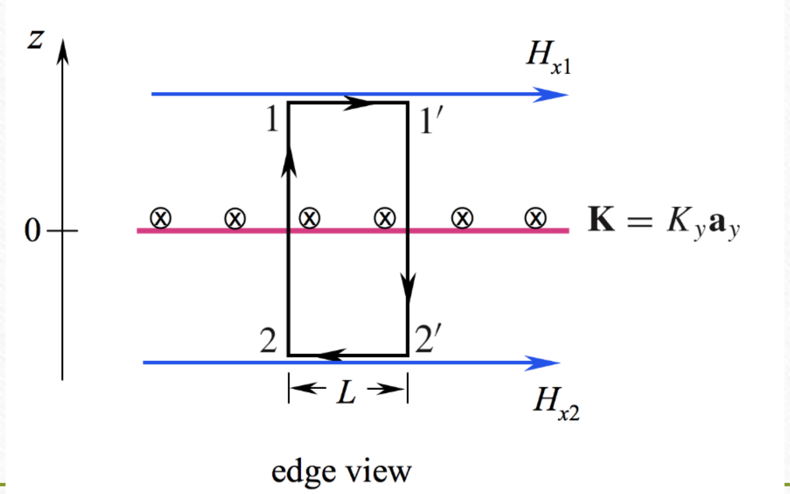
Magnetic Field Arising from a Current Sheet

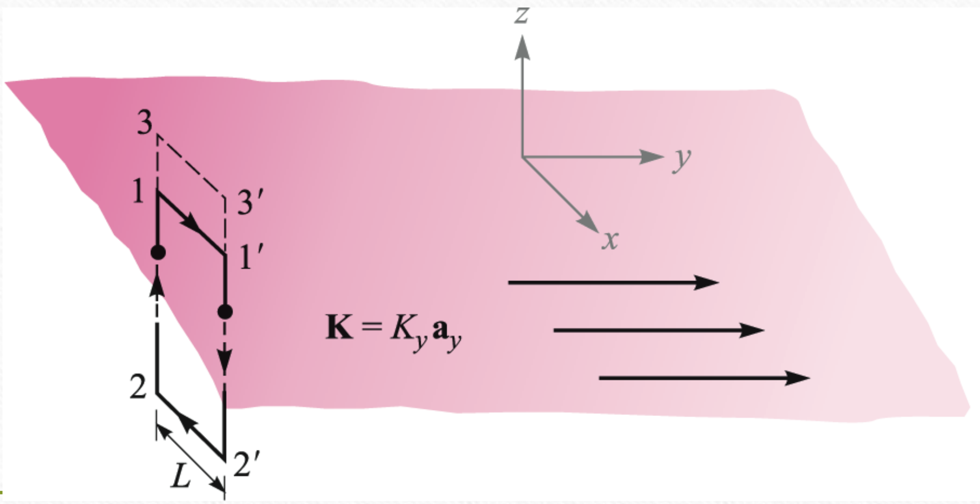
For a uniform plane current in the y direction, we expect an x-directed **H** field from symmetry.

Applying Ampere’s circuital law to the path 1-1′- 2′– 2- 1, we find:

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad \underline{H_{x1} - H_{x2} = K_y}$$

In other words, the magnetic field is discontinuous across the current sheet by the magnitude of the surface current density.



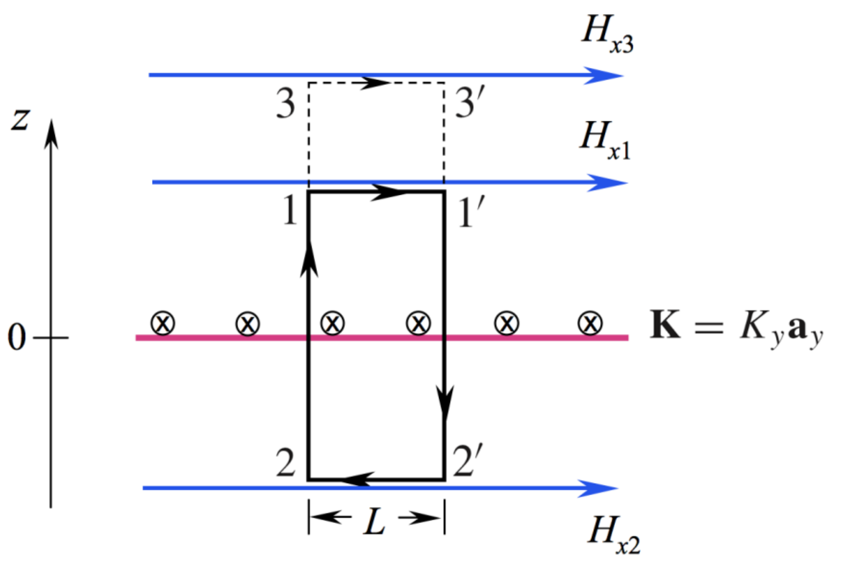


Magnetic Field Arising from a Current Sheet

If instead, the upper path is elevated to the line between **3** and **3 ′**, the same current is enclosed and we would have

$H_{x3} - H_{x2} = K_y$ from which we conclude that $H_{x3} = H_{x1}$

so the field is constant in each region (above and below the current plane)



edge view

Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. we may state that

$$H_x = \tfrac{1}{2} K_y \quad (z > 0)$$

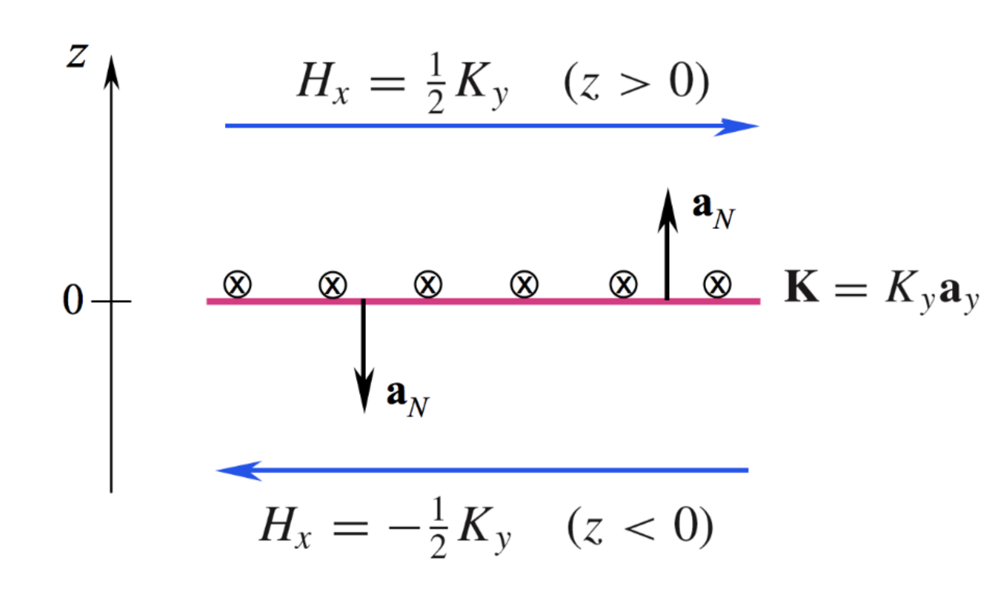
and

$$H_x = -\tfrac{1}{2} K_y \quad (z < 0)$$

Magnetic Field Arising from a Current Sheet

The actual field configuration is shown below, in which magnetic field above the current sheet is equal in magnitude, but in the direction opposite to the field below the sheet.

The field in either region is found by the cross product:

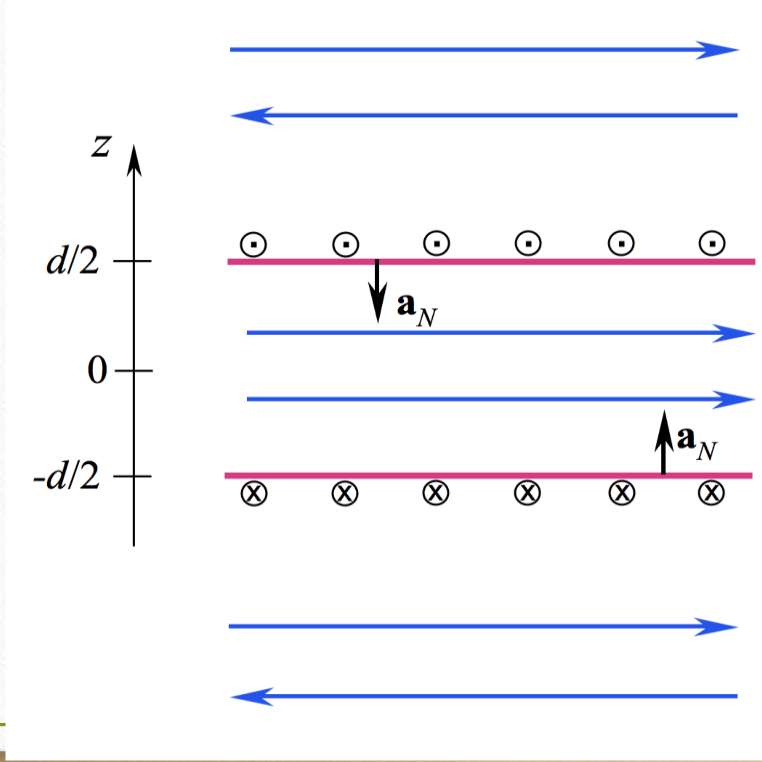


$$\boxed{\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N}$$

where $\mathbf{a}_N$ is the unit vector that is normal to the current sheet, and that points into the region in which the magnetic field is to be evaluated.

Magnetic Field Arising from Two Current Sheets

Here are two parallel currents, equal and opposite, as you would find in a parallel-plate transmission line. If the sheets are much wider than their spacing, then the magnetic field will be contained in the region between plates, and will be nearly zero outside.



$H_{x2}\,(z > d/2\,)$
$H_{x1}\,(z > d/2\,)$

These fields cancel for current sheets of infinite width.

$\mathbf{K}_1 = -K_y\,\mathbf{a}_y$

$H_{x1}\,(-d\,/2 < z < d/2\,)$
$H_{x2}\,(-d\,/2 < z < d/2\,)$

These fields are equal and add to give

$$\boxed{\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (-d/2 < z < d/2\,)} \qquad (12)$$

$\mathbf{K}_2 = K_y\,\mathbf{a}_y$

where **K** is either $\mathbf{K}_1$ or $\mathbf{K}_2$

$H_{x1}\,(z < -d/2\,)$
$H_{x2}\,(z < -d/2\,)$

These fields cancel for current sheets of infinite width.

Home Work

D8.3. Express the value of $\mathbf{H}$ in rectangular components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

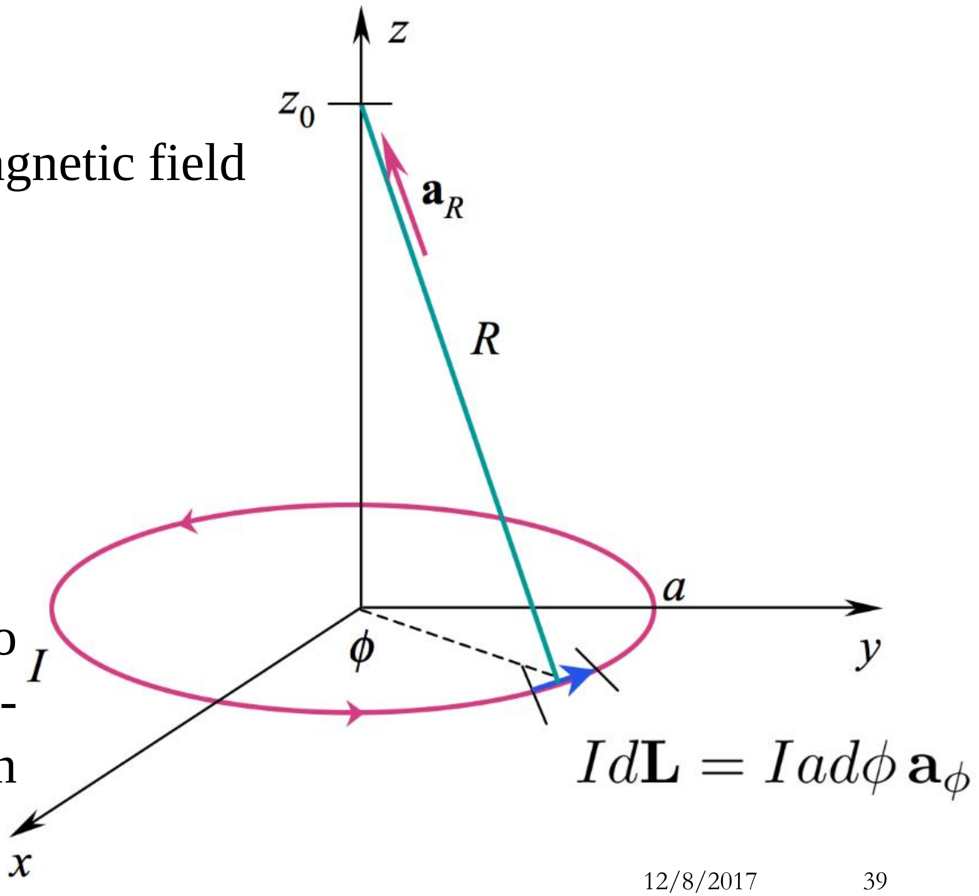
Magnetic Field of a Solenoid

Current Loop Field

Using the Biot-Savart Law, we previously found the magnetic field on the z axis from a circular current loop:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

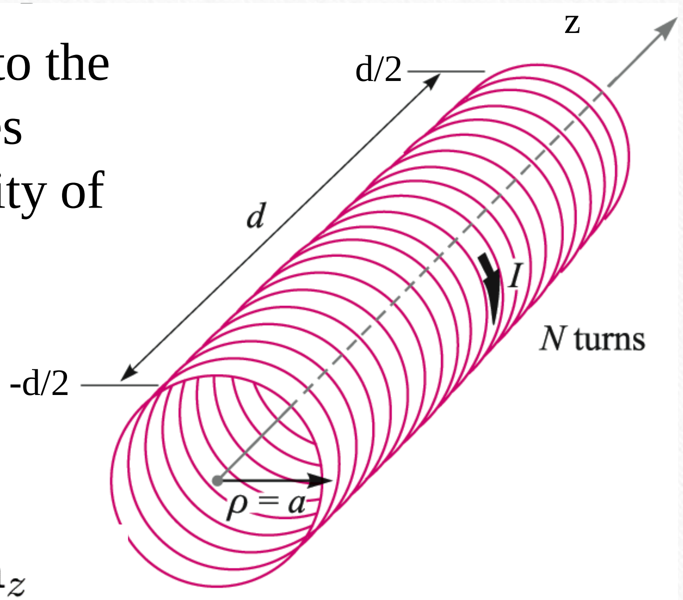
We will now use this result as a building block to construct the magnetic field on the axis of a solenoid - formed by a stack of identical current loops, centered on the z axis.



Magnetic Field of a Solenoid

Magnetic Field inside a real Solenoid

We consider the single current loop field as a differential contribution to the total field from a stack of N closely-spaced loops, each of which carries current I. The length of the stack (solenoid) is d, so therefore the density of turns will be N/d.



Now the current in the turns within a differential length, dz, will be

$$\boxed{dI = \frac{N}{d} I dz}$$

We consider this as our differential "loop current"

so that the previous result for **H** from a single loop: $\mathbf{H} = \dfrac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$

now becomes:

$$d\mathbf{H} = \frac{(N/d) I dz (\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}} \qquad (13)$$

in which z is measured from the center of the coil, where we wish to evaluate the field.

Magnetic Field of a Solenoid

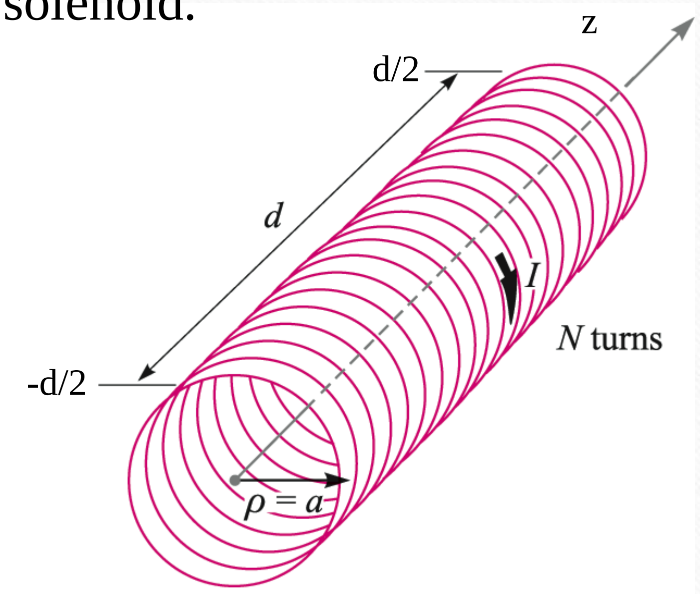
Magnetic Field inside a real Solenoid

The total field on the z axis at z = 0 will be the sum of the field contributions from all turns in the coil -- or the integral of d**H** over the length of the solenoid.

$$\mathbf{H} = \int d\mathbf{H} = \int_{-d/2}^{d/2} \frac{(N/d)I\,dz(\pi a^2)\mathbf{a}_z}{2\pi(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \int_{-d/2}^{d/2} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \frac{d}{a^2\sqrt{a^2+(d/2)^2}} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2+(d/2)^2}}$$



Magnetic Field of a Solenoid

Magnetic Field inside a real Solenoid

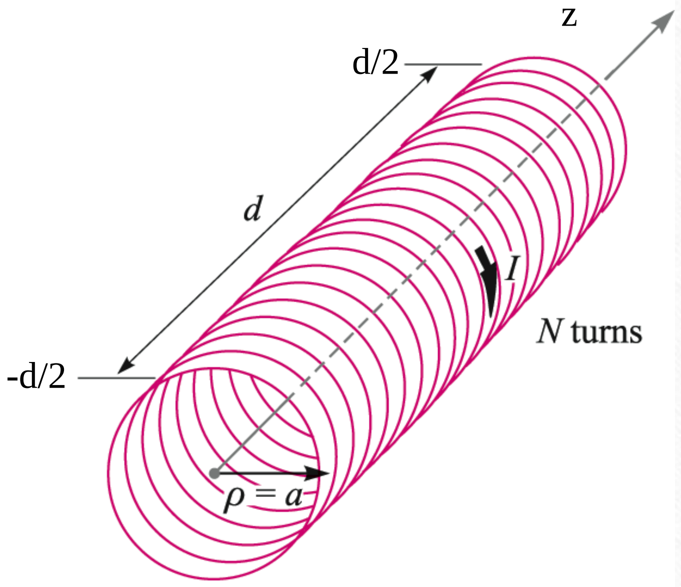
Approximation for Long Solenoids

We now have the on-axis field at the solenoid midpoint (z = 0):

$$\mathbf{H} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}} \qquad (14)$$

Note that for long solenoids, for which $d \gg a$, the result simplifies to:

$$\boxed{\mathbf{H} \doteq \frac{NI}{d}\,\mathbf{a}_z \quad (d >> a)} \qquad (15)$$

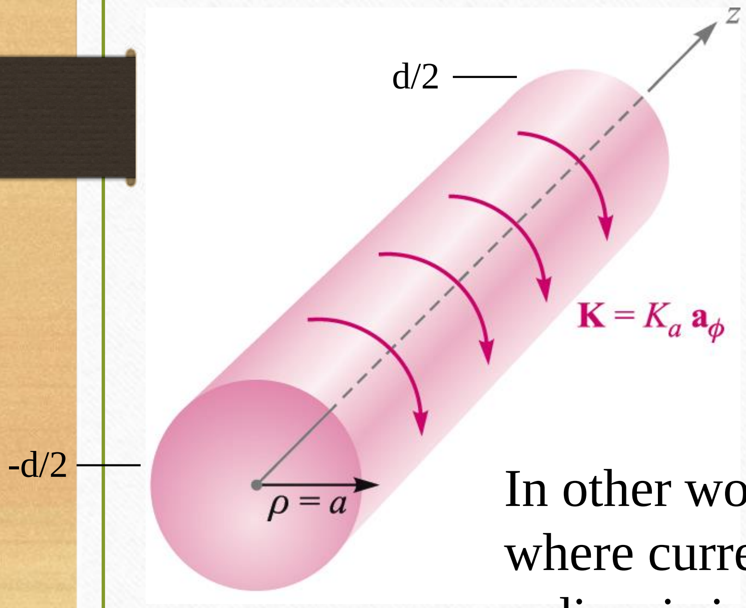


This result is valid at all on-axis positions deep within long coils -- at distances from each end of several radii.

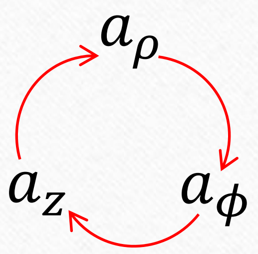
Magnetic Field of a Solenoid

Another Interpretation: Continuous Surface Current

The solenoid of our previous example was assumed to have many tightly-wound turns, with several existing within a differential length, dz. We could model such a current configuration as a continuous surface current of density $\mathbf{K} = K_a\,\mathbf{a}_\phi$ A/m.



From Eq. (12) $\because \mathrm{H} = \mathrm{K} \times a_n$



For a point inside the solenoid $a_n = -a_\rho$

$$\therefore \mathrm{H} = (K_a a_\phi) \times (-a_\rho) = K_a a_z$$

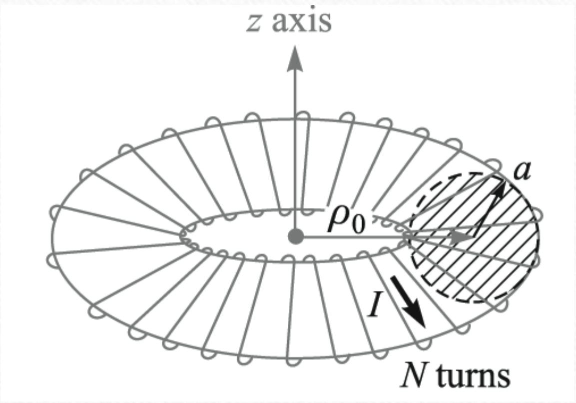
Since $\mathbf{K} = K_a\,\mathbf{a}_\phi = \dfrac{NI}{d}\,\mathbf{a}_\phi \quad \mathrm{A/m}$ $\Rightarrow$ $\mathrm{H} = K_a a_z = \dfrac{NI}{d} a_z$

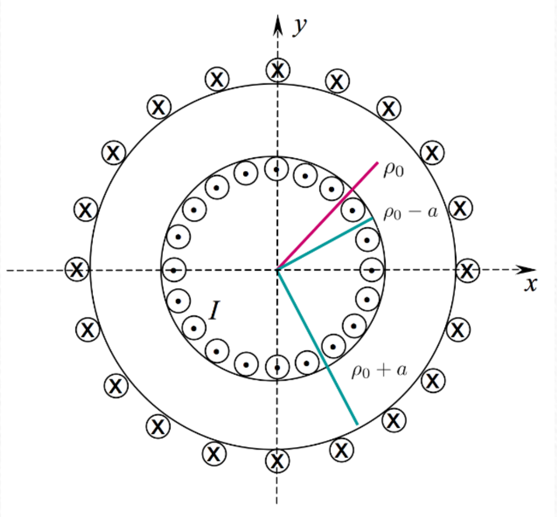
In other words, the on-axis field magnitude near the center of a cylindrical current sheet, where current circulates around the z axis, and whose length is much greater than its radius, is just the surface current density.

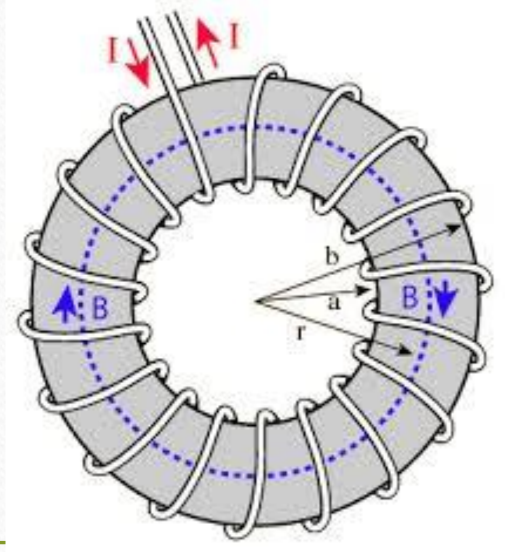
Toroid Magnetic Field

A toroid is a doughnut-shaped set of windings around a core material. The cross-section could be circular (as shown here, with radius a) or any other shape.

Below, a slice of the toroid is shown, with current emerging from the screen around the inner periphery (in the positive z direction). The windings are modeled as N individual current loops, each of which carries current I.







Ampere's Law as Applied to a Toroid

Ampere's Circuital Law can be applied to a toroid by taking a closed loop integral around the circular contour C at radius ρ. Magnetic field **H** is presumed to be circular, and a function of radius only at locations within the toroid that are not too close to the individual windings. Under this condition, we would assume: $\mathbf{H} = H_\phi \, \mathbf{a}_\phi$

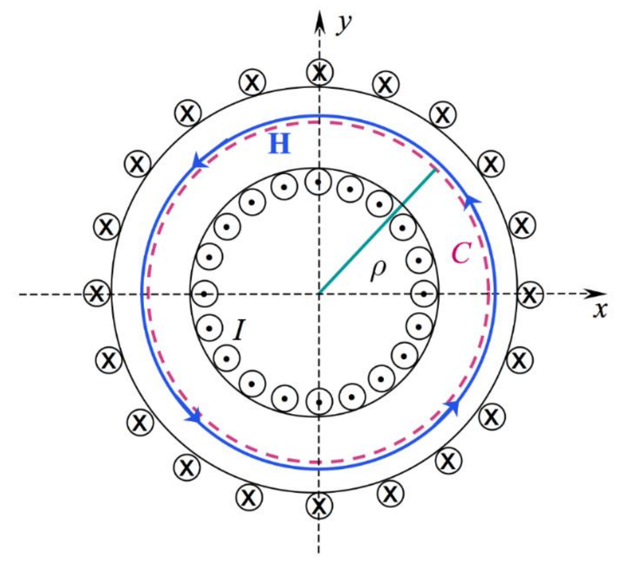
This approximation improves as the density of turns gets higher (using more turns with finer wire).

Ampere's Law now takes the form:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = I_{encl} = NI$$

so that….

$$\boxed{H_\phi = \frac{NI}{2\pi\rho} \quad (\rho_0 - a < \rho < \rho_0 + a)}$$



Performing the same integrals over contours drawn in the outside region will lead to zero magnetic field there, because no current is enclosed in either case.

Report

Idea of Magnetic Tape Recording